

# Thermodynamics of third order Lovelock adS black holes in the presence of Born-Infeld type nonlinear electrodynamics

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In this paper, we obtain topological black hole solutions of third order Lovelock gravity couple with two classes of Born-Infeld type nonlinear electrodynamics with anti-de Sitter asymptotic structure. We investigate geometric and thermodynamics properties of the solutions and obtain conserved quantities of the black holes. We examine the first law of thermodynamics and find that the conserved and thermodynamic quantities of the black hole solutions satisfy the first law of thermodynamics. Finally, we calculate the heat capacity and determinant of Hessian matrix to evaluate thermal stability in both canonical and grand canonical ensembles. Moreover, we consider extended phase space thermodynamics to obtain generalized first law of thermodynamics as well as extended Smarr formula.

## I. INTRODUCTION

Regarding string theory and brane world cosmology, it has been shown that spacetimes possess more than four dimensions. Taking into account higher dimensional spacetimes, we know that the general conserved symmetric tensor that depends on the metric and its derivatives up to second order is not the Einstein tensor.

One of the natural generalization of Einstein theory in higher dimensional spacetimes, in which contains most of Einstein assumptions, is Lovelock gravity [1]. Lovelock gravity field equation contains metric derivatives no higher second order and therefore its quantization is free of ghost [2]. Since the higher curvature terms of Lovelock Lagrangian appear in the low-energy limit of string theory, black hole solutions of Lovelock gravity have been attracting renewed interest. The action of Lovelock gravity in a compressed form can be written as

$$I_G = \int d^d x \sqrt{-g} \sum_{k=0}^{d/2} \alpha_k \mathcal{L}_k, \quad (1)$$

where  $\alpha_k$ 's are arbitrary constants and  $\mathcal{L}_k$ 's are the Euler densities of the  $2k$ -dimensional manifolds with the following explicit form

$$\mathcal{L}_k = \delta_{\rho_1 \sigma_1 \dots \rho_k \sigma_k}^{\mu_1 \nu_1 \dots \mu_k \nu_k} R_{\mu_1 \nu_1}^{\rho_1 \sigma_1} \dots R_{\mu_k \nu_k}^{\rho_k \sigma_k}. \quad (2)$$

In Eq. (2),  $\delta_{\rho_1 \sigma_1 \dots \rho_k \sigma_k}^{\mu_1 \nu_1 \dots \mu_k \nu_k}$  and  $R_{\mu\nu}^{\rho\sigma}$  are, respectively, the generalized totally anti-symmetric Kronecker delta and the Riemann tensor.

The objective of this paper is to find topological (asymptotically adS) black hole solutions of third order Lovelock gravity in the presence of two classes of Born-Infeld (BI) type nonlinear electrodynamics (NED). Some black object solutions of third order Lovelock theory coupled with NED have been studied before [3]. Recently, one of us considered BI type Lagrangians to obtain the black hole solutions [4, 5]. The Lagrangians of exponential and logarithmic forms of BI type theories may be defined as

$$\mathcal{L}(\mathcal{F}) = \begin{cases} \beta^2 \left( \exp\left(-\frac{\mathcal{F}}{\beta^2}\right) - 1 \right), & \text{ENED} \\ -8\beta^2 \ln\left(1 + \frac{\mathcal{F}}{8\beta^2}\right), & \text{LNED} \end{cases}, \quad (3)$$

where  $\beta$  is called the nonlinearity parameter, the Maxwell invariant is  $\mathcal{F} = F_{\mu\nu} F^{\mu\nu}$  in which  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is the Faraday tensor and  $A_\mu$  is the gauge potential. Although BI type models was introduced with various motivations, the important motivation of considering the BI type NED theories comes from the fact that these theories may be originated if one regards the loop corrections [6]. Taking into account the coupling of BI type theories with Einstein and Gauss-Bonnet gravity, it was shown that although there are some analogies between the BI type theories, there exist some differences between them as well [4, 5, 7, 8]. Now, we would like to obtain new topological black hole

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solutions of the mentioned models of BI type theories coupled with third order Lovelock gravity and investigate their geometric and thermodynamic properties.

The outline of the paper is as follows. We present the topological black hole solutions in Sec. II. Sec. III, is devoted to investigate conserved and thermodynamic quantities of topological black holes. We also analyze the thermodynamic stability of the solutions in the canonical and grand canonical ensembles. We finish our paper with some conclusions.

## II. TOPOLOGICAL ADS BLACK HOLES IN THIRD ORDER LOVELOCK GRAVITY

The gravitational and electromagnetic field equations of third order Lovelock gravity in the presence of NED may be written as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha_2 \mathcal{G}_{\mu\nu} + \alpha_3 \mathcal{H}_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \mathcal{L}(\mathcal{F}) - 2F_{\mu\lambda} F_{\nu}^{\lambda} \mathcal{L}_{\mathcal{F}}, \quad (4)$$

$$\partial_{\mu} (\sqrt{-g} \mathcal{L}_{\mathcal{F}} F^{\mu\nu}) = 0, \quad (5)$$

where  $G_{\mu\nu}$  is the Einstein tensor,  $\mathcal{G}_{\mu\nu}$  and  $\mathcal{H}_{\mu\nu}$  are, respectively, the second and third orders Lovelock tensor given as

$$\mathcal{G}_{\mu\nu} = 2(R_{\mu\sigma\kappa\tau} R_{\nu}^{\sigma\kappa\tau} - 2R_{\mu\rho\nu\sigma} R^{\rho\sigma} - 2R_{\mu\sigma} R^{\sigma}_{\nu} + R R_{\mu\nu}) - \frac{1}{2} g_{\mu\nu} \mathcal{L}_2, \quad (6)$$

$$\begin{aligned} \mathcal{H}_{\mu\nu} = & -3[4R^{\tau\rho\sigma\kappa} R_{\sigma\kappa\lambda\rho} R^{\lambda}_{\nu\tau\mu} - 8R^{\tau\rho}_{\lambda\sigma} R^{\sigma\kappa}_{\tau\mu} R^{\lambda}_{\nu\rho\kappa} + 2R_{\nu}^{\tau\sigma\kappa} R_{\sigma\kappa\lambda\rho} R^{\lambda\rho}_{\tau\mu} \\ & - R^{\tau\rho\sigma\kappa} R_{\sigma\kappa\tau\rho} R_{\nu\mu} + 8R^{\tau}_{\nu\sigma\rho} R^{\sigma\kappa}_{\tau\mu} R^{\rho}_{\kappa} + 8R^{\sigma}_{\nu\tau\kappa} R^{\tau\rho}_{\sigma\mu} R^{\kappa}_{\rho} \\ & + 4R_{\nu}^{\tau\sigma\kappa} R_{\sigma\kappa\mu\rho} R^{\rho}_{\tau} - 4R_{\nu}^{\tau\sigma\kappa} R_{\sigma\kappa\tau\rho} R^{\rho}_{\mu} + 4R^{\tau\rho\sigma\kappa} R_{\sigma\kappa\tau\mu} R_{\nu\rho} + 2R R_{\nu}^{\kappa\tau\rho} R_{\tau\rho\kappa\mu} \\ & + 8R^{\tau}_{\nu\mu\rho} R^{\rho}_{\sigma} R^{\sigma}_{\tau} - 8R^{\sigma}_{\nu\tau\rho} R^{\tau}_{\sigma} R^{\rho}_{\mu} - 8R^{\tau\rho}_{\sigma\mu} R^{\sigma}_{\tau} R_{\nu\rho} - 4R R^{\tau}_{\nu\mu\rho} R^{\rho}_{\tau} \\ & + 4R^{\tau\rho} R_{\rho\tau} R_{\nu\mu} - 8R^{\tau}_{\nu} R_{\tau\rho} R^{\rho}_{\mu} + 4R R_{\nu\rho} R^{\rho}_{\mu} - R^2 R_{\nu\mu}] - \frac{1}{2} g_{\mu\nu} \mathcal{L}_3 \end{aligned} \quad (7)$$

In the recent equations,  $\mathcal{L}_2$  and  $\mathcal{L}_3$  denote the Gauss-Bonnet Lagrangian and the third order Lovelock term, given as

$$\mathcal{L}_2 = R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4R_{\mu\nu} R^{\mu\nu} + R^2, \quad (8)$$

$$\begin{aligned} \mathcal{L}_3 = & 2R^{\mu\nu\sigma\kappa} R_{\sigma\kappa\rho\tau} R^{\rho\tau}_{\mu\nu} + 8R^{\mu\nu}_{\sigma\rho} R^{\sigma\kappa}_{\nu\tau} R^{\rho\tau}_{\mu\kappa} + 24R^{\mu\nu\sigma\kappa} R_{\sigma\kappa\nu\rho} R^{\rho}_{\mu} \\ & + 3R R^{\mu\nu\sigma\kappa} R_{\sigma\kappa\mu\nu} - 12R R_{\mu\nu} R^{\mu\nu} + 24R^{\mu\nu\sigma\kappa} R_{\sigma\mu} R_{\kappa\nu} + 16R^{\mu\nu} R_{\nu\sigma} R^{\sigma}_{\mu} + R^3. \end{aligned} \quad (9)$$

In addition,  $\alpha_i$ 's are Lovelock coefficients and  $\mathcal{L}_{\mathcal{F}} = \frac{d\mathcal{L}(\mathcal{F})}{d\mathcal{F}}$ . Now, we consider the following line element to obtain the  $(n+1)$ -dimensional static topological black hole solutions:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\check{g}^2 \quad (10)$$

where  $d\check{g}^2$  is the metric of an  $(n-1)$ -dimensional hypersurface with constant curvature  $(n-1)(n-2)k$  and volume  $V_{n-1}$  with the following explicit form

$$d\check{g}^2 = \begin{cases} d\theta_1^2 + \sum_{i=2}^{n-1} \prod_{j=1}^{i-1} \sin^2 \theta_j d\theta_i^2 & k = 1 \\ d\theta_1^2 + \sinh^2 \theta_1 d\theta_2^2 + \sinh^2 \theta_1 \sum_{i=3}^{n-1} \prod_{j=2}^{i-1} \sin^2 \theta_j d\theta_i^2 & k = -1 \\ \sum_{i=1}^{n-1} d\phi_i^2 & k = 0 \end{cases} \quad (11)$$

Since the boundary of these spacetimes may be positive, zero or negative constant curvature, these metric is usually called topological spacetime. At first we consider the electromagnetic equation (5), to obtain the nonzero component of the gauge potential

$$A_{\mu} = \delta_{\mu}^0 \times \begin{cases} -\frac{\beta r \sqrt{L_W}}{2(n-2)(3n-4)} ((n-1) \zeta L_W + 3n-4), & \text{ENED} \\ \frac{2\beta^2 r^n}{nq} (\eta-1), & \text{LNED} \end{cases}, \quad (12)$$

where  $q$  is an integration constant which is related to the electric charge and

$$L_W = \text{Lambert}W\left(\frac{4q^2}{\beta^2 r^{2d-4}}\right) \quad (13)$$

$$\zeta = {}_2F_1\left([1], \left[\frac{5n-6}{2(n-1)}\right], \frac{L_W}{2(n-1)}\right), \quad (14)$$

$$\eta = {}_2F_1\left(\left[-\frac{1}{2}, \frac{-n}{2(n-1)}\right], \left[\frac{n-2}{2(n-1)}\right], 1-\Gamma^2\right). \quad (15)$$

$$\Gamma = \sqrt{1 + \frac{q^2}{\beta^2 r^{2(n-1)}}} \quad (16)$$

It was shown that the mentioned gauge potential reduce to that of Maxwell field for weak field limit  $\beta \rightarrow \infty$ . Considering a special case  $\alpha_3 = \frac{\alpha^2}{3(n-2)(n-3)(n-4)(n-5)}$  and  $\alpha_2 = \frac{\alpha}{(n-2)(n-3)}$ , we can show that the metric function

$$f(r) = k + \frac{r^2}{\alpha} \left(1 - H^{1/3}\right), \quad (17)$$

with

$$H = 1 + \frac{3\alpha m}{r^n} + \frac{6\alpha\Lambda}{n(n-1)} + \frac{3\alpha\beta^2}{n(n-1)r^n} \times \begin{cases} r^n + \frac{2nq}{\beta} \int \left(\sqrt{Lw} - \frac{1}{\sqrt{Lw}}\right) dr, & \text{ENED} \\ -8r^n + 8n \int r^{n-1} [\Gamma - \ln(\frac{\Gamma+1}{2})] dr, & \text{LNED} \end{cases}, \quad (18)$$

satisfies all components of the field equations (4). The parameter  $m$  is an integration constant which is related to finite mass as [9]

$$M = \frac{V_{n-1}(n-1)m}{16\pi}. \quad (19)$$

We should note that since computing the total mass leads to an infinite quantity, one may solve this problem by using of background subtraction method whose asymptotical geometry matches that of the solutions. Another approach comes from the fact that adding an additional surface action does not alter the bulk equations of motion. It is known as AdS/CFT inspired counterterm method [10]. All methods have the same result and one may obtain the finite mass (19) (see appendix for more details).

Now, we should discuss the existence of singularity(ies). To do so, it is usual to calculate the Kretschmann scalar. It is easy to find that the Kretschmann is

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = f''^2(r) + 2(n-1)\left(\frac{f'(r)}{r}\right)^2 + 2(n-1)(n-2)\left(\frac{f(r)-k}{r^2}\right)^2. \quad (20)$$

Taking into account the metric function with Eq. (20), one finds the Kretschmann diverges at  $r = 0$  and is finite for  $r \neq 0$ . In order to interpret the curvature singularity as a black hole, we should look for the horizon. The horizon(s) is (are) located at the root(s) of  $g^{rr} = f(r) = 0$ . Numerical calculations shows that, depending on the values of  $\alpha$  and  $\beta$ , the metric function has two real positive roots, one extreme root, one non-extreme root or it may be positive definite (for more details see [4, 5]). Hence obtained solutions may be interpreted as the black holes with two horizons, extreme black holes, Schwarzschild-like black holes (one non-extreme horizon) or naked singularity. Moreover, using the series expansion of metric function for large  $r$ , one finds

$$\begin{aligned} f(r)|_{\text{large } r} &= k - \frac{2\Lambda[n(n-1) - 2\alpha\Lambda]r^2}{n^2(n-1)^2} - \frac{m[n(n-1) - 4\alpha\Lambda]}{n(n-1)r^{n-2}} + \frac{2q^2[n(n-1) - 4\alpha\Lambda]}{n(n-1)^2(n-2)r^{2n-4}} + \frac{\alpha m^2}{r^{2n-2}} \\ &\quad - \frac{4\alpha m q^2}{(n-1)(n-2)r^{3n-4}} + \frac{4\alpha q^4}{(n-1)^2(n-2)^2r^{4n-6}} - \frac{4[n(n-1) - 4\alpha\Lambda]q^4}{\Upsilon n(3n-4)\beta^2 r^{4n-6}} + O\left(\frac{1}{r^{5n-6}}\right), \\ \Upsilon &= \begin{cases} 2(n-1)^2 & \text{ENED} \\ n^2 & \text{LNED} \end{cases}. \end{aligned} \quad (21)$$

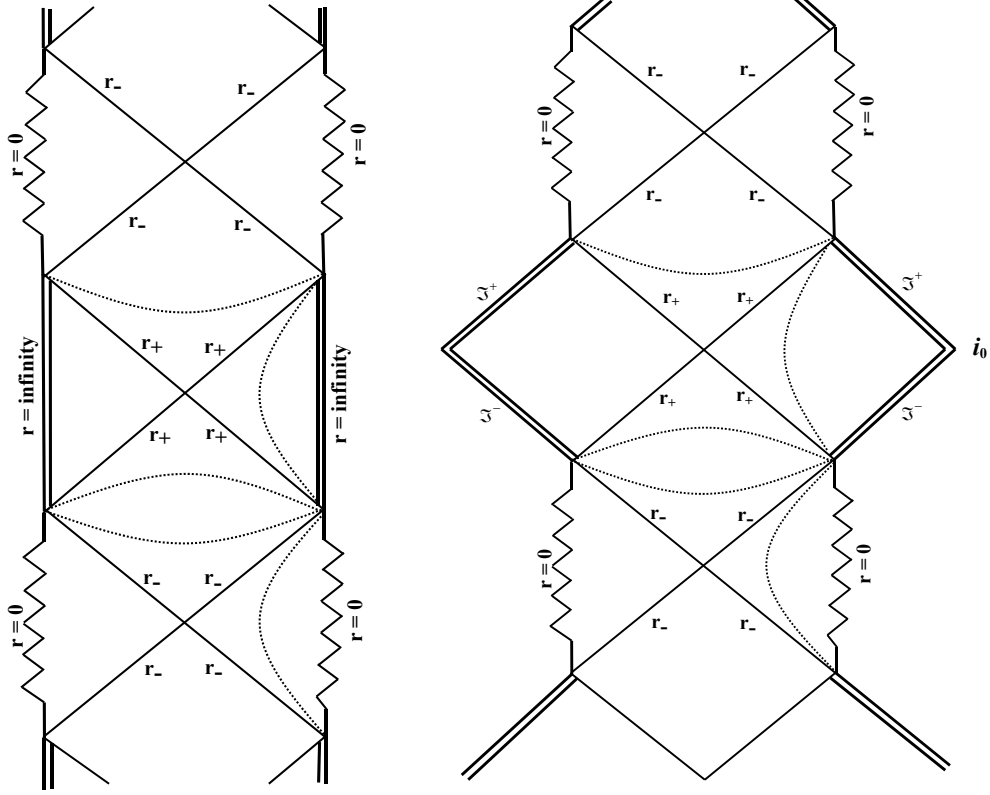


FIG. 1: Carter-Penrose diagram for the asymptotically adS (left figure) and the asymptotically flat (right figure) black holes when the metric function has two real positive roots ( $r_-$  and  $r_+$ ). "dotted curves represent  $r = \text{constant}$ "

Eq. (21) shows that the second term is dominant for large  $r$  in which confirms that these black holes are asymptotically adS if we replace  $\Lambda$  with  $\Lambda_{eff} = \frac{\Lambda[n(n-1)-2\alpha\Lambda]}{n(n-1)}$ . In other words, Lovelock gravity may modify the cosmological constant and, as we expect,  $\Lambda_{eff} \rightarrow \Lambda$  for vanishing  $\alpha$ .

In order to provide additional information for the conformal structure of the solutions, we can use the conformal compactification method to draw the Carter-Penrose (conformal) diagram (see Figs. 1-3 for more details). The Carter-Penrose diagrams confirm that the singularity may be timelike (such as Reissner-Nordström black holes) or spacelike (such as Schwarzschild black holes). In other words, depending the values of the nonlinearity parameter, one can find that  $\lim_{r \rightarrow 0} f(r)$  can be positive or negative (for more details we refer the reader to Ref. [4, 5]). Drawing the Carter-Penrose diagrams shows that the causal structure of the solutions are asymptotically well behaved.

### III. THERMODYNAMIC PROPERTIES AND THERMAL STABILITY

In this section, we calculate the conserved and thermodynamic quantities, and check the first law of black hole thermodynamics. Then we perform the stability criterion.

At first, we apply the definition of surface gravity to obtain the Hawking temperature

$$T = \frac{1}{2\pi} \sqrt{-\frac{1}{2} (\nabla_\mu \chi_\nu) (\nabla^\mu \chi^\nu)}, \quad (22)$$

where  $\chi$  is the temporal Killing vector,  $\partial_t$ . One obtains

$$T = \frac{f'(r_+)}{4\pi} = \frac{(n-1)k [3(n-2)r_+^4 + 3(n-4)k\alpha r_+^2 + (n-6)\alpha^2] - 6\Lambda r_+^6 + 3\beta^2 r_+^6 \Psi}{12\pi(n-1)r_+(r_+^2 + k\alpha)^2}, \quad (23)$$

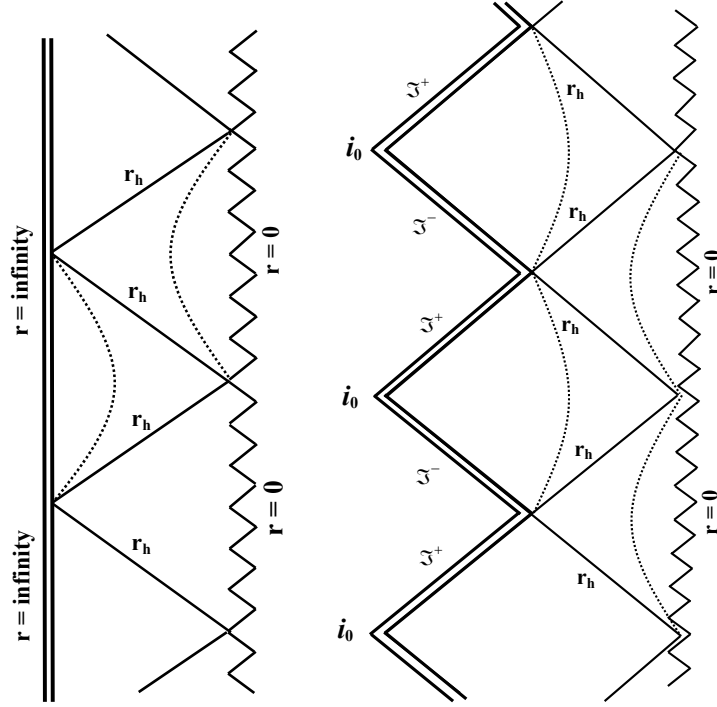


FIG. 2: Carter-Penrose diagram for the asymptotically adS (left figure) and the asymptotically flat (right figure) black holes when the metric function has one real positive extreme root ( $r_- = r_+ = r_h$ ) (extreme black hole). "dotted curves represent  $r = \text{constant}$ "

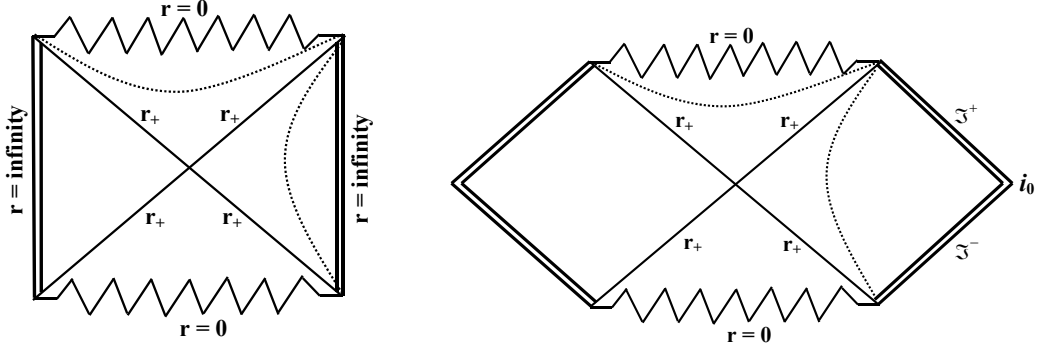


FIG. 3: Carter-Penrose diagram for the asymptotically adS (left figure) and the asymptotically flat (right figure) black holes when the metric function has one real positive root ( $r_+$ ) (the same as Schwarzschild black hole). "dotted curves represent  $r = \text{constant}$ "

where

$$\Psi = \begin{cases} \frac{2(1-Lw_+)}{\sqrt{Lw_+}} \sqrt{\Gamma_+^2 - 1} - 1, & \text{ENED} \\ 8 \left[ 1 + \ln \left( \frac{1+\Gamma_+}{2} \right) - \Gamma_+ \right], & \text{LNED} \end{cases}, \quad (24)$$

$$\Gamma_+ = \sqrt{1 + \frac{q^2}{\beta^2 r_+^{2(n-1)}}}, \quad (25)$$

which shows that the temperature depends on the Lovelock parameter as well as nonlinearity factor of electrodynamics.

Now, we calculate the entropy of the black hole solutions. Since we regard the Lovelock gravity, the area law of the Black hole entropy does not satisfy in general [11]. The expression of the entropy of Lovelock black holes may be derived by Hamiltonian formalism [12] (for its generalization for arbitrarily high order derivatives of the curvature, see [13]), yielding

$$S = \frac{V_{n-1}}{4} r_+^{n-1} \left( 1 + \frac{2(n-1)k}{(n-3)} \frac{\alpha}{r_+^2} + \frac{(n-1)k^2}{(n-5)} \frac{\alpha^2}{r_+^4} \right), \quad (26)$$

It is clear that Eq. (26) reproduces the area law for Einstein gravity ( $\alpha \rightarrow 0$ ).

In order to obtain the electric charge, we calculate the flux of the electromagnetic field at infinity. It is easy to show that

$$Q = \frac{V_{n-1}}{4\pi} q. \quad (27)$$

Eq. (27) confirms that the electric charge does not depend on the nonlinearity parameter.

In order to calculate the electric potential of the black holes, one should consider a reference. Considering nonzero component of the gauge potential (or the electric field), one finds that for  $r \rightarrow \infty$ , both the gauge potential and the electric field vanishes. Therefore, it is natural to calculate the electric potential of the event horizon of black holes,  $r_+$ , with respect to the infinity as reference [14]. We obtain

$$\begin{aligned} \Phi &= A_\mu \chi^\mu|_{r \rightarrow \infty} - A_\mu \chi^\mu|_{r=r_+} = \\ &\begin{cases} \frac{\beta r_+ \sqrt{Lw_+}}{2(n-2)(3n-4)} [(n-1)\zeta_+ L_{W+} + 3n-4], & \text{ENED} \\ -\frac{2\beta^2 r_+^n}{nq} (\eta_+ - 1), & \text{LNED} \end{cases}. \end{aligned} \quad (28)$$

$$\zeta_+ = {}_2F_1 \left( [1], \left[ \frac{5n-6}{2(n-1)} \right], \frac{L_{W+}}{2(n-1)} \right), \quad (29)$$

$$\eta_+ = {}_2F_1 \left( \left[ -\frac{1}{2}, \frac{-n}{2(n-1)} \right], \left[ \frac{n-2}{2(n-1)} \right], 1 - \Gamma_+^2 \right), \quad (30)$$

Here, we are in a position to check the first law of thermodynamics for various horizon topology. At first we obtain the finite mass  $M$  as a function of the entropy and electric charge as the extensive quantities. Straightforward calculations show that

$$M(S, Q) = \frac{(n-1)r_+^n}{48\pi\alpha} \left[ \left( 1 + \frac{k\alpha}{r_+^2} \right)^3 - 1 \right] - \frac{\Lambda r_+^n}{8\pi n} - \Theta, \quad (31)$$

where

$$\Theta = \begin{cases} \frac{\beta^2 r_+^n}{16\pi n} + \frac{\beta q}{8\pi} \int \left( \sqrt{Lw} - \frac{1}{\sqrt{Lw}} \right) dr \Big|_{r_+}, & \text{ENED} \\ -\frac{\beta^2 r_+^n}{2\pi n} + \frac{\beta^2}{2\pi} \int r^{n-1} [\Gamma - \ln(\frac{\Gamma+1}{2})] dr \Big|_{r_+}, & \text{LNED} \end{cases}. \quad (32)$$

Now, we use the first law to define temperature and electric potential as the intensive parameters conjugate to the entropy and electric charge

$$T = \left( \frac{\partial M}{\partial S} \right)_Q = \frac{\left( \frac{\partial M}{\partial r_+} \right)_Q}{\left( \frac{\partial S}{\partial r_+} \right)_Q}, \quad (33)$$

$$\Phi = \left( \frac{\partial M}{\partial Q} \right)_S = \frac{\left( \frac{\partial M}{\partial q} \right)_{r_+}}{\left( \frac{\partial Q}{\partial q} \right)_{r_+}}. \quad (34)$$

Numerical analysis shows that Eqs. (33) and (34) are equal to Eqs. (23) and (28), respectively, and therefore we deduce that these quantities satisfy the first law of thermodynamics

$$dM = TdS + \Phi dQ. \quad (35)$$

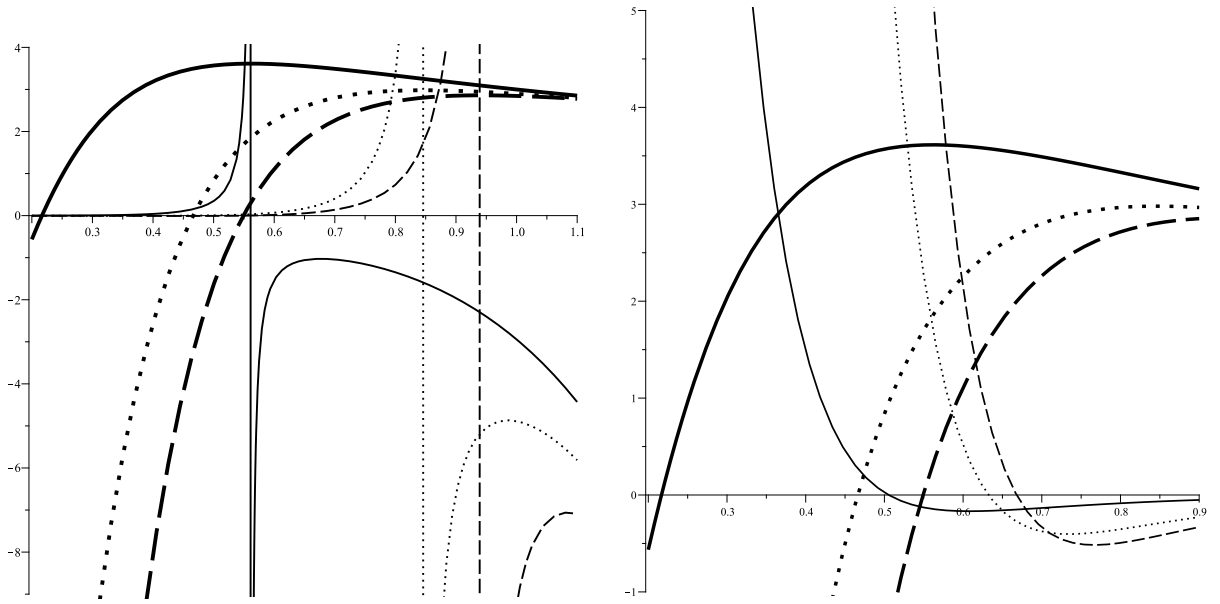


FIG. 4:  $C_Q$  (left),  $H$  (right) diagrams versus  $r_+$  for  $k = 1$ ,  $n = 6$ ,  $q = 1$ ,  $\alpha = 0.1$ , and  $\beta = 0.05$  (solid line)  $\beta = 0.5$  (dotted line)  $\beta = 1$  (dashed line) "bold lines are related to corresponding temperatures"

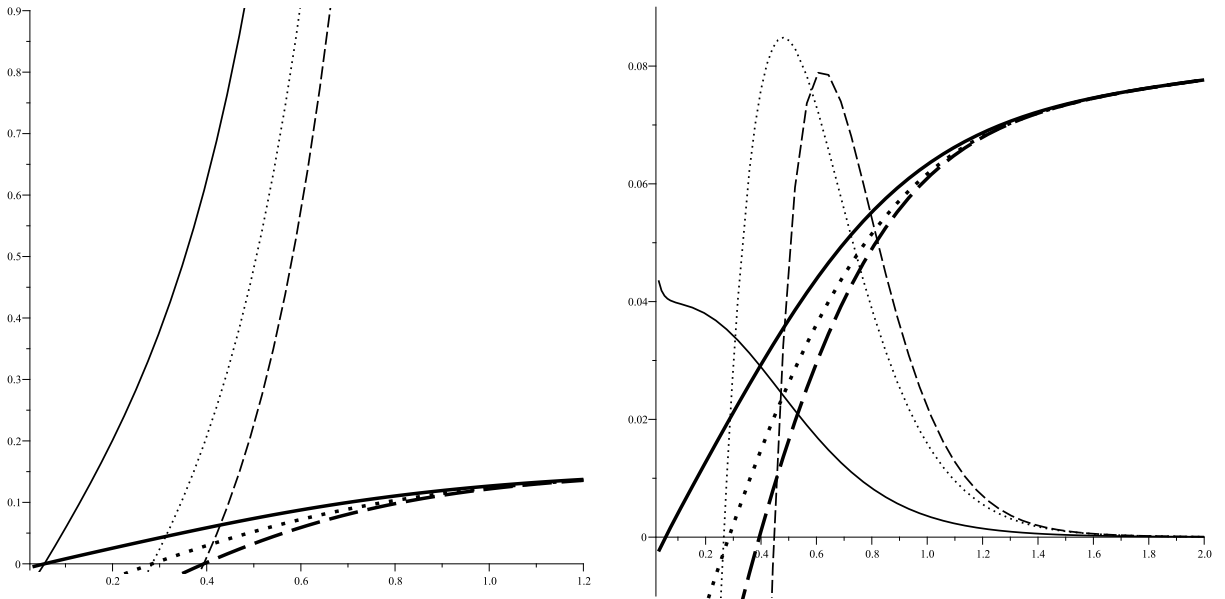


FIG. 5:  $C_Q$  (left),  $H$  (right) diagrams versus  $r_+$  for  $k = 1$ ,  $n = 6$ ,  $q = 1$ ,  $\alpha = 1$ , and  $\beta = 0.05$  (solid line)  $\beta = 0.5$  (dotted line)  $\beta = 1$  (dashed line) "bold lines are related to corresponding temperatures"

### A. Thermal stability

In order to discuss the thermal stability conditions, one may use both canonical and grand canonical ensembles. The positivity of the heat capacity and determinant of the Hessian matrix are the requirements usually referred to stability criterion in canonical and grand canonical ensembles, respectively. Since  $M$  is a function of  $S$  and  $Q$ , we can write the heat capacity and determinant of the Hessian matrix with the following explicit forms

$$C_Q = \frac{T}{\left(\frac{\partial^2 M}{\partial S^2}\right)_Q}, \quad (36)$$

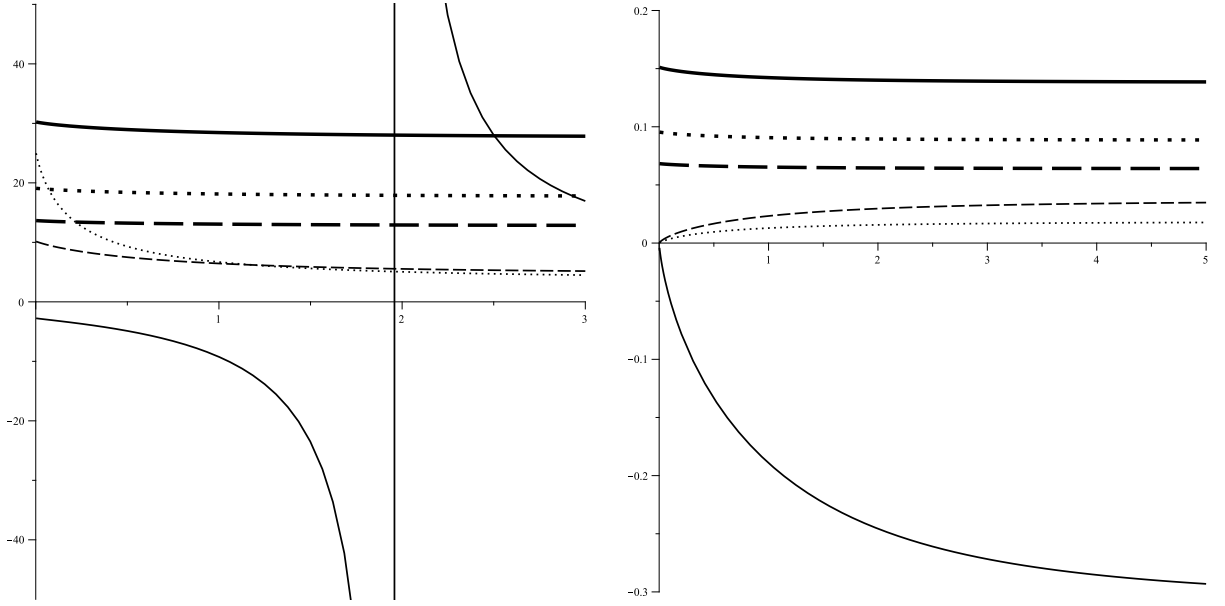


FIG. 6:  $C_Q$  (left),  $H$  (right) diagrams versus  $\beta$  for  $k = 1$ ,  $n = 6$ ,  $q = 1$ ,  $r_+ = 1$ , and  $\alpha = 0.1$  (solid line)  $\alpha = 0.5$  (dotted line)  $\alpha = 0.9$  (dashed line) **bold lines are related to corresponding temperatures**

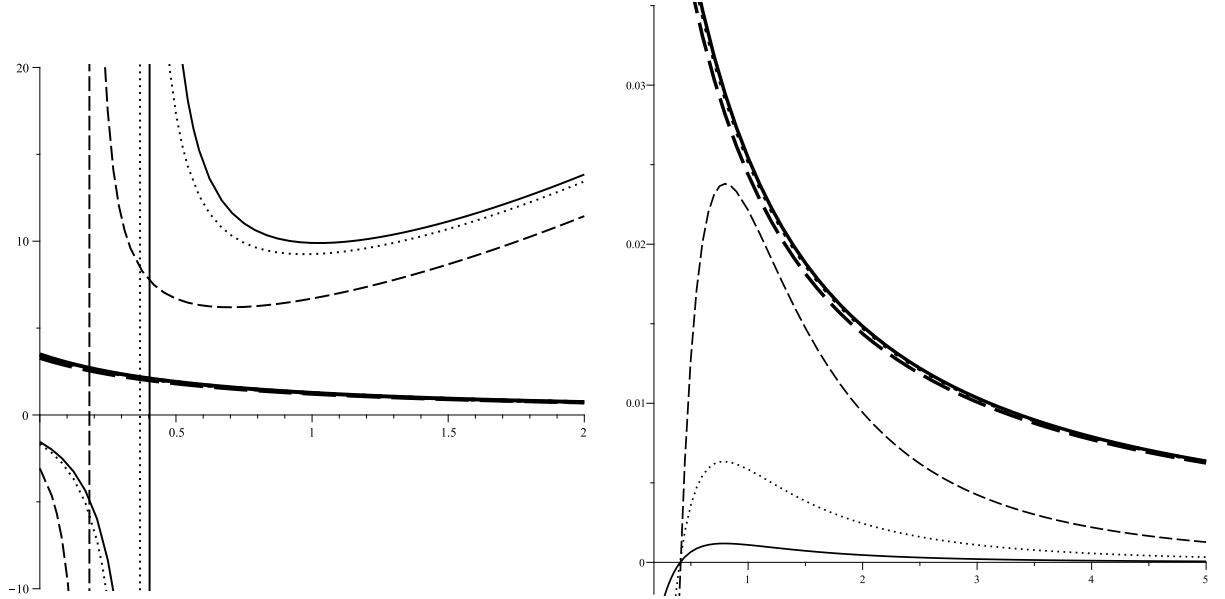


FIG. 7:  $C_Q$  (left),  $H$  (right) diagrams versus  $\alpha$  for  $k = 1$ ,  $n = 6$ ,  $q = 1$ ,  $r_+ = 1$ , and  $\beta = 0.05$  (solid line)  $\beta = 0.5$  (dotted line)  $\beta = 1$  (dashed line) **bold lines are related to corresponding temperatures**

$$H = \det \begin{bmatrix} \frac{\partial^2 M}{\partial S^2} & \frac{\partial^2 M}{\partial S \partial Q} \\ \frac{\partial^2 M}{\partial Q \partial S} & \frac{\partial^2 M}{\partial Q^2} \end{bmatrix}. \quad (37)$$

Analytical calculations of the heat capacity and determinant of the Hessian matrix are too large and therefore we leave out the analytical result for reasons of economy. We plot some figures to discuss the stability conditions. Numerical calculations show that although large black holes have positive definite temperature, there is a lower limit for the horizon radius of physical small black holes,  $r_0$ . It is notable that  $r_0$  increases for increasing  $\beta$  (decreasing  $\alpha$ ). In addition, we find that, for small values of  $\alpha$  ( $\alpha < \alpha_c$ ), there are two  $r_a$  and  $r_b$  ( $r_a < r_b$ ), in which the black holes are stable for  $r_0 < r_+ < r_a$  and  $r_+ > r_b$  (see Fig. 4 and also following three tables). Moreover, for large values of  $\alpha$  ( $\alpha > \alpha_c$ ) and  $r_+ > r_c$  the black holes are stable. We should note that for canonical ensemble one finds  $r_c = r_0$  and



for grand canonical ensemble  $r_c > r_0$  (see Fig. 5). Figs. 6 and 7 confirm that, regardless of value of  $\beta$ , we encounter an unstable phase for  $\alpha < \alpha_c$ .

$r_+ =$	0.5	1	4	400
$C_Q =$	0.35	-2.9	5148	$1.2 \times 10^{13}$
$H =$	0.02	-0.03	$4.8 \times 10^{-5}$	$1.2 \times 10^{-20}$

Table (1): corresponding to Fig. 4 for  $\beta = 0.05$ .

$r_+ =$	0.6	1	4	400
$C_Q =$	0.07	-4.8	5148	$1.2 \times 10^{13}$
$H =$	0.51	-0.13	$4.8 \times 10^{-5}$	$1.2 \times 10^{-20}$

Table (2): corresponding to Fig. 4 for  $\beta = 0.5$ .

$r_+ =$	0.6	1	4	400
$C_Q =$	0.02	-9.25	5148	$1.2 \times 10^{13}$
$H =$	2.1	-0.19	$4.8 \times 10^{-5}$	$1.2 \times 10^{-20}$

Table (3): corresponding to Fig. 4 for  $\beta = 1$ .

#### IV. EXTENDED PHASE SPACE AND SMARR FORMULA

In previous section we considered the usual discussions of thermodynamic properties of asymptotically adS black holes, in which the cosmological constant is treated as a fixed parameter. However, there are some motivations to view the cosmological constant as a variable (for e.g. see [15]). In addition, there exist some theories where some physical constants such as gauge coupling constants, Newton constant, Lovelock coefficients and BI parameter may not be fixed values but dynamical ones. In that case, it is natural to consider these variable parameters into the first law of black hole thermodynamics [16]. Considering the cosmological constant as a thermodynamic pressure, the black hole mass  $M$  should be explained as enthalpy rather than internal energy of the system [17]. In the geometric units, one can identify the cosmological constant with the pressure as

$$P = -\frac{\Lambda}{8\pi}, \quad (38)$$

where the thermodynamic quantity conjugate to the pressure is called thermodynamic volume of black holes. In addition, it was shown that the Smarr formula may be extended to Lovelock gravity as well as nonlinear theories of electrodynamics [18].

Geometrical techniques (scaling argument) were used to derive an extension of the first law and its related modified Smarr formula that includes variations in the cosmological constant, Lovelock coefficient and also nonlinearity parameter. In our case, Lovelock gravity in the presence of the NED,  $M$  should be the function of entropy, pressure, charge, Lovelock parameter and BI coupling coefficient [18]. Regarding the previous section, we find that those thermodynamic quantities satisfy the following differential form

$$dM = TdS + \Phi dQ + VdP + \mathcal{A}'_1 d\alpha_2 + \mathcal{A}'_2 d\alpha_3 + \mathcal{B}d\beta. \quad (39)$$

where we have achieved  $T$  and  $\Phi$ , and one can obtain

$$\begin{aligned} V &= \left( \frac{\partial M}{\partial P} \right)_{S, Q, \alpha_2, \alpha_3, \beta}, \\ \mathcal{A}'_1 &= \left( \frac{\partial M}{\partial \alpha_2} \right)_{S, Q, P, \alpha_3, \beta}, \\ \mathcal{A}'_2 &= \left( \frac{\partial M}{\partial \alpha_3} \right)_{S, Q, P, \alpha_2, \beta}, \\ \mathcal{B} &= \left( \frac{\partial M}{\partial \beta} \right)_{S, Q, P, \alpha_2, \alpha_3}. \end{aligned}$$

Using the redefinition of  $\alpha_2$  and  $\alpha_3$  with respect to the single parameter,  $\alpha$ , we can rewrite  $\mathcal{A}'_1 d\alpha_2 + \mathcal{A}'_2 d\alpha_3$  as a single differential form

$$\begin{aligned} d\alpha_2 &= \frac{1}{(n-2)(n-3)} d\alpha, \\ d\alpha_3 &= \frac{2\alpha}{3(n-2)(n-3)(n-4)(n-5)} d\alpha. \end{aligned}$$

Moreover, by scaling argument, we can obtain the generalized Smarr relation for our asymptotically adS solutions in the extended phase space

$$(d-3)M = (d-2)TS + (d-3)Q\Phi - 2PV + 2(\mathcal{A}_1\alpha + \mathcal{A}_2\alpha^2) - \mathcal{B}\beta \quad (40)$$

where

$$\begin{aligned} V &= \frac{r_+^n}{n}, \\ \mathcal{A}_1 &= \frac{(n-1)k^2 r_+^{n-4}}{16\pi} - \frac{(n-1)kTr_+^{n-3}}{2(n-3)}, \\ \mathcal{A}_2 &= \frac{(n-1)k^3 r_+^{n-6}}{24\pi} - \frac{(n-1)k^2 Tr_+^{n-5}}{2(n-5)}, \\ \mathcal{B}_{|_{\text{ENED}}} &= \frac{q(n-1)r_+(L_{W+})^{\frac{3}{2}} F\left([1], \left[\frac{5n-6}{2n-2}, \frac{L_{W+}}{2n-2}\right]\right)}{8\pi n(3n-4)} - \frac{\beta r_+^n}{8\pi n} + \frac{q\beta r_+^{n+1} \sqrt{L_{W+}}(1-L_{W+})}{8\pi n(1+L_{W+})} + \frac{2qr_+}{8\pi n \sqrt{L_{W+}}(1+L_{W+})}, \\ \mathcal{B}_{|_{\text{LNED}}} &= \frac{\beta r_+^n}{2\pi n^2} \left[ -(n-1)(1-\Gamma_+^2) F\left(\left[\frac{1}{2}, \frac{n-2}{2n-2}\right], \left[\frac{3n-4}{2n-2}, 1-\Gamma_+^2\right]\right) + 2n \ln\left(\frac{1+\Gamma_+}{2}\right) + (3n-2)(1-\Gamma_+) \right]. \end{aligned}$$

Regarding the mentioned argument and using Eqs. (38) and (23), one can obtain the equation of state  $P(V, T)$  to compare the black hole system with the Van der Waals fluid equation in  $(n+1)$ -dimensions [19].

## V. CLOSING REMARKS

In this paper we considered third order Lovelock gravity in the presence of exponential and logarithmic forms of NED models. Regardless of naked singularities, we obtained topological black hole solutions with two horizons or one (non-)extreme horizon. We found that replacing  $\Lambda$  with an effective cosmological constant,  $\Lambda_{eff}$ , one may obtain asymptotically adS solutions. In other words, Lovelock gravity and also BI type NED models do not alter the asymptotical behavior of the solutions. We obtained thermodynamics and conserved quantities of the topological black holes and found that the Lovelock gravity does not affect the temperature, entropy and finite mass only for black holes with Ricci flat horizon,  $k=0$ . Moreover, we showed that the thermodynamics and conserved quantities satisfy the first law of thermodynamics.

We performed stability criterion in both canonical and grand canonical ensembles by use of numerical analysis only for  $k=1$ . We found a lower bound for the horizon radius,  $r_0 \geq 0$ , in which the temperature is positive for  $r_+ > r_0$ . We showed that the nonlinearity parameter,  $\beta$ , and also Lovelock coefficient,  $\alpha$  can affect the value of  $r_0$ . Then we studied the heat capacity and determinant of Hessian matrix and showed that for  $\alpha < \alpha_c$ , there are two limits  $r_a$  and  $r_b$  ( $r_a < r_b$ ), in which the black holes have an unstable phase for  $r_a < r_+ < r_b$ . Furthermore, we found an lower limit ( $r_c$ ) in which for  $\alpha > \alpha_c$  and  $r_+ > r_c$  the black holes are stable. In addition, we found that for canonical ensemble one finds  $r_c = r_0$  and for grand canonical ensemble  $r_c > r_0$ . Calculations showed that regardless of value of values of  $\beta$ , there is an unstable phase of black hole solutions for  $\alpha < \alpha_c$ .

At last, we have discussed the extended phase space in which the cosmological constant, nonlinearity and Lovelock parameters considered as dynamical variables. We have calculated generalized Smarr formula and also modified first law of thermodynamics. Extended phase space help us to investigate the similarities between the thermodynamical behavior of black hole system under studied and the Van der Waals gas/liquid system.

Finally, we should note that for the sake of economy, we investigated stability conditions only for  $k=1$ . One may regard other horizon topology for discussion of thermal stability. In addition, it is worthwhile to mention that it would be interesting to investigate the phase transition by Geometrothermodynamics approach [20]. In addition, one can follow the section IV to discuss about the concept of extended phase space thermodynamics and  $P-V$  criticality of the Lovelock black holes with BI type NED [19]. We leave these problems to our forthcoming independent works.

### Acknowledgments

We would like to thank the anonymous referee for useful suggestions and enlightening comments. The authors wish to thank Shiraz University Research Council. This work has been supported financially by Center for Excellence in Astronomy & Astrophysics of Iran (CEAAI-RIAAM).

### Appendix

The action of third order Lovelock gravity in the presence of NED which is related to the field equations (4) and (5) is

$$\mathcal{I}_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} [R - 2\Lambda + \alpha_2 \mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \mathcal{L}(\mathcal{F})] + \mathcal{I}_b, \quad (41)$$

where  $\mathcal{L}_2$ ,  $\mathcal{L}_3$  and  $\mathcal{L}(\mathcal{F})$  were defined before. The last term in Eq. 41 is boundary action. The integral of Eq. 41 does not have a well-defined variational principle, since one encounters a total derivative that produces a surface integral involving the derivative of  $\delta g_{\mu\nu}$  normal to the boundary. The normal derivative terms do not vanish by themselves, but are cancelled by the variation of the suitable surface term (Gibbons-Hawking-York boundary term [21, 22]) with the following explicit form

$$\mathcal{I}_b = -\frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-\gamma} [K + \alpha_2 L_{2b} + \alpha_3 L_{3b}], \quad (42)$$

with

$$L_{2b} = 2 \left( J - 2\hat{G}_{ab}^{(1)} K^{ab} \right), \quad (43)$$

$$L_{3b} = 3(P - 2\hat{G}_{ab}^{(2)} K^{ab} - 12\hat{R}_{ab} J^{ab} + 2\hat{R}J - 4K\hat{R}_{abcd}K^{ac}K^{bd} - 8\hat{R}_{abcd}K^{ac}K_e^b K^{ed}), \quad (44)$$

where  $\gamma_{\mu\nu}$  and  $K$  are, respectively, the induced metric and the trace of extrinsic curvature of boundary,  $\hat{G}_{ab}^{(1)}$  and  $\hat{G}_{ab}^{(2)}$  denote the  $n$ -dimensional Einstein and second order Lovelock tensors of the metric  $\gamma_{ab}$  while  $J$  and  $P$  are the traces of

$$J_{ab} = \frac{1}{3}(2KK_{ac}K_b^c + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^{cd}K_{db} - K^2K_{ab}), \quad (45)$$

and

$$\begin{aligned} P_{ab} = & \frac{1}{5} \{ [K^4 - 6K^2K^{cd}K_{cd} + 8KK_{cd}K_e^d K^{ec} - 6K_{cd}K^{de}K_{ef}K^{fc} + 3(K_{cd}K^{cd})^2] K_{ab} \\ & - (4K^3 - 12KK_{ed}K^{ed} + 8K_{de}K_f^e K^{fd}) K_{ac}K_b^c - 24KK_{ac}K^{cd}K_{de}K_b^e \\ & + 12(K^2 - K_{ef}K^{ef}) K_{ac}K^{cd}K_{db} + 24K_{ac}K^{cd}K_{de}K^{ef}K_{bf} \}. \end{aligned} \quad (46)$$

In general the action  $\mathcal{I}_G$ , the Hamiltonian and other associated conserved quantities diverge when evaluated on the solutions. Due to the fact that our spacetime is asymptotically adS, one can use the systematic method to regulate the gravitational action of asymptotically adS solutions which is through the use of the counterterm method. It was shown that the counterterm approach become quite reasonable when applied to AdS/CFT, as the boundary counterterm has a natural interpretation as conventional field theory counterterm that show up in the dual CFT [23].

The counterterm action is a functional of the boundary curvature invariants and do not affect on the symmetries and field equations of the bulk  $\mathcal{M}$

$$I_{ct} = \int_{\partial\mathcal{M}} d^n x \sqrt{-h} L(l, \hat{R}, \nabla \hat{R}, \dots). \quad (47)$$

In a general manner, the counterterm in Lovelock gravity is a scalar constructed from the curvature invariants of the boundary as in the case of Einstein gravity [24, 25]. Although one can use the procedure of Ref. [25] to compute the counterterm action for arbitrary horizon topology, for the sake of brevity and simplification, we deal with the spacetime with zero curvature boundary ( $\hat{R}_{abcd}(\gamma) = 0$ ). In this case all the counterterm containing the curvature invariants of the boundary are zero (see [3, 26] for more details) and the counterterm reduces to

$$\mathcal{I}_{ct} = \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-\gamma} \left( \frac{n-1}{l_{\text{eff}}} \right), \quad (48)$$

where  $l_{\text{eff}}$  is given by

$$l_{\text{eff}} = \frac{15\sqrt{\alpha \left[1 - \left(1 - \frac{3\alpha}{l^2}\right)^{1/3}\right]}}{9 \left(1 + \frac{\alpha}{l^2}\right) - \left[2 + \left(1 - \frac{3\alpha}{l^2}\right)^{1/3}\right]^2}, \quad (49)$$

It is notable that the effective  $l_{\text{eff}}$  reduces to  $l$  as  $\alpha$  goes to zero. Having the finite action and using the Brown–York method of a quasilocal definition [27] with Eq. (41)-(48), one can introduce a divergence-free stress-energy tensor as follows

$$T^{ab} = \frac{2}{\sqrt{-\gamma}} \frac{\partial(\mathcal{I}_G + \mathcal{I}_{\text{ct}})}{\partial \gamma_{ab}} = \frac{1}{8\pi} \left[ (K^{ab} - K\gamma^{ab}) + 2\alpha(3J^{ab} - J\gamma^{ab}) + \frac{n-1}{l_{\text{eff}}} \gamma^{ab} \right]. \quad (50)$$

The quasilocal conserved quantities associated with the stress-energy tensor of Eq. (50) can be defined as

$$Q(\xi) = \int_B d^{n-1}\varphi \sqrt{\gamma} T_{ab} n^a \xi^b, \quad (51)$$

where the timelike unit vector  $n^a$  is normal to the boundary  $B$  and  $\xi^b$  is the Killing vector. Regarding temporal Killing vector  $\xi = \partial/\partial t$  and taking into account Eqs. (51) and (50), we can calculate the mass per unit volume  $V_{n-1}$  as

$$M = \frac{(n-1)}{16\pi} m. \quad (52)$$

We should note that the parameter  $m$  can be calculated by using of the fact that the metric function vanishes at the event horizon,  $r_+$ . Although one can check that the form of Eq. (52) is valid for  $k = \pm 1, 0$ , we should indicate that, unlike  $k = \pm 1$  cases, the mass parameter,  $m$ , does not depend on the Lovelock parameter for the boundary flat solutions.

Although we used the counterterm method to calculate the finite mass, one may find different methods in the literature for computing the finite mass. It will be interesting to study the conditions that enable those prescriptions to provide the right mass for the solutions obtained here.

One of the best known prescriptions is that of Arnowitt-Deser-Misner (ADM), which can be most applied in asymptotically flat spacetimes. In addition, the ADM method may also be applied to asymptotically anti-de Sitter space [9]. In such case, the mass may be extracted by comparison to a suitable reference background (e.g. vacuum adS). Furthermore, we refer the reader to the Ashtekar-Magnon-Das (AMD) formula [28], the Hamiltonian method of Regge and Teitelboim [29], the generalized Komar integral of Lovelock gravity [30] and the subtraction method of Brown and York [27, 31].

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